The effects of radiative transfer on turbulent flow of a stratified fluid

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SUMMARY

Assuming local thermodynamic equilibrium in the fluid, an expression is derived for the rate of destruction of the mean square of the temperature fluctuations by radiative transfer of heat. This takes a particularly simple form (*a*) if the fluid is effectively transparent over distances equal to the scale of the turbulent motion, when the effect appears as a decay time for temperature fluctuations from the mean, and (*b*) if the fluid is effectively opaque, when the effect is of an increased conductivity due to radiation. A theory of the interaction of the temperature and velocity fields developed in a previous paper shows that, if the radiative effects are relatively weak, a sudden collapse of the turbulent motion occurs while the flux Richardson number is still less than one. If the radiative effects are strong, the turbulent intensity approaches zero as the flux Richardson number approaches one. The effects of radiation are always to increase the critical value of the ordinary Richardson number. Criteria for fully turbulent motion of an unrestricted flow are given in terms of the gradients of mean velocity and mean temperature and of the rate of radiative cooling. The relevance of these calculations to motions of the atmosphere is briefly discussed.

1. INTRODUCTION

In the past decade, evidence that the upper atmosphere between seventy and one hundred kilometres above sea-level may be in turbulent motion has been accumulating. Most of the evidence is indirect and based on the irregular fading of radio echoes from the ionosphere (e.g. Briggs, Phillips & Shinn 1950), but Liller & Whipple (1954, Vol. 1, p. 112) have made observations of luminous meteor trails which show the existence of irregular velocity gradients resembling closely those found in turbulent flows. The nearly constant composition of the atmosphere from sea-level to these heights also shows that mixing on the molar scale must occur sufficiently frequently to counteract diffusive separation of the atmospheric components, and these mixing motions might be turbulent motions.

As most of this evidence of turbulent motion is indirect, it is useful to ask whether our present knowledge of turbulence would lead us to expect turbulent flow with the gradients of mean velocity and temperature occurring in this region. The criterion for turbulent motion of air of sea-level density is usually expressed as a critical value of the Richardson number and, although there is considerable disagreement over the critical value, a simple application of this criterion to the upper atmosphere makes turbulent motion appear very unlikely*. However, the derivation of this criterion neglects radiative transfer of heat, a process which is known to dominate the heat balance in the upper atmosphere, and its inclusion will cause a reduction in the magnitude of the buoyancy forces which are responsible for the inhibition of turbulent motion. In general, radiative transfer tends to destroy the effects of density stratification caused by temperature gradients as Goody (1956) has shown in his investigation of cellular convection between parallel horizontal planes. The purpose of this paper is to discuss the turbulent motion in a stratified fluid with appreciable radiative transfer of heat using a similar analysis to that used by the author in a previous discussion of stratified flow with negligible radiative transfer, and to derive criteria for the maintenance of turbulent motion.

2. Turbulent transfer rates in stratified flow

In a recent paper (Townsend 1958), the interactions between the fields of velocity and temperature in a stratified flow were considered, and relations were found between the levels of the turbulent fluctuations and transport rates and the gradients of mean velocity and mean temperature. These relations form the basis of the present paper but, before quoting them, it may be useful to set out briefly the steps in their derivation.

We consider the steady flow of a nearly perfect gas, for example air containing only small proportions of water-vapour, carbon dioxide, ozone, etc., with velocity variations small compared with the local velocity of sound, with a length scale small compared with the scale height of the atmosphere, with temperature variations small compared with the absolute temperature, and unaffected by the rotation of the earth. Without serious loss of generality, we may suppose the mean flow to be nearly unidirectional and horizontal with an appreciable gradient only in the vertical direction, e.g. a horizontal mixing layer between two streams, and describe the flow in the usual coordinate system with Ox in the horizontal direction of flow and with Oz vertically upward. Then,

U, W are the components of the mean velocity parallel to Ox, Oz respectively,

u, v, w are the components of the velocity fluctuation parallel to Ox, Oy, Oz respectively,

T is the mean absolute temperature,

* For example, for a zero lapse rate and a wind shear of 10 metre sec^{-1} km^{-1} the Richardson number is 3·5.

θ is the local temperature fluctuation,
ν is the kinematic viscosity,
κ is the thermometric conductivity (heat diffusivity),
ρ is the mean density,
c_p is the specific heat at constant pressure,
g is the downward acceleration due to gravity.

To the approximation implied by the restrictions set out above, the equation for the kinetic energy of the turbulent velocity fluctuations* is

$$\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial t} + \overline{uw}\frac{\partial U}{\partial z} + U\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial x} + W\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial z} + \frac{\partial}{\partial z}\left(\frac{1}{2}\overline{q^2 w} + \frac{1}{\rho}\overline{pw}\right) = \frac{g}{T}\overline{\theta w} - \epsilon, \tag{2.1}$$

where $q^2 = u^2+v^2+w^2$ and ϵ is the rate at which turbulent kinetic energy is being converted to heat by the action of viscosity. A second equation of physical importance is the equation for the mean square of the temperature fluctuation. It is

$$\frac{\partial(\frac{1}{2}\overline{\theta^2})}{\partial t} + \overline{\theta w}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right) + U\frac{\partial(\frac{1}{2}\overline{\theta^2})}{\partial x} + W\frac{\partial(\frac{1}{2}\overline{\theta^2})}{\partial z} + \frac{\partial}{\partial z}(\tfrac{1}{2}\overline{\theta^2 w}) = \kappa\overline{\theta\nabla^2\theta} - \beta\overline{\theta^2}, \tag{2.2}$$

where $-\beta\overline{\theta^2} = \overline{\mathscr{R}\theta}(\rho c_p)^{-1}$ and $\mathscr{R}$ is the net rate of gain of heat per unit volume by radiative exchange. These two equations are, respectively, direct deductions from the equations of motion and from the equation for the internal energy (heat equation).

To simplify the form of these equations, we make use of the observation that, in free turbulent flows remote from solid boundaries, there is a sharply defined surface separating turbulent fluid from the surrounding undisturbed fluid and that the turbulent fluid is remarkably homogeneous in intensity and scale over any section of the flow (Townsend 1956). This means that there is a physical meaning in using averaged values of the turbulent intensity and of the mean square temperature fluctuation, defined as

$$\frac{1}{D}\int_{-\infty}^{\infty}\overline{q^2}\,dz \quad \text{and} \quad \frac{1}{D}\int_{-\infty}^{\infty}\overline{\theta^2}\,dz,$$

where D is the mean vertical extent of the fully turbulent fluid. If equations (2.1) and (2.2) are averaged in this way, they become

$$\overline{uw}\frac{\partial U}{\partial z} + \frac{\partial}{\partial x}(\tfrac{1}{2}U\overline{q^2}) = \frac{g}{T}\overline{\theta w} - \epsilon \tag{2.3}$$

and

$$\overline{\theta w}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right) + \frac{\partial}{\partial x}(\tfrac{1}{2}U\overline{\theta^2}) = \kappa\overline{\theta\nabla^2\theta} - \beta\overline{\theta^2}, \tag{2.4}$$

where all the quantities are to be understood as suitably averaged values.

* In a steady flow, the time derivatives of equations (2.1) and (2.2) are zero. They are included to make clearer the meaning of the equations.

We also neglect the terms representing gain or loss through advection, i.e. $\partial/\partial x(\frac{1}{2}U\overline{q^2})$ and $\partial/\partial x(\frac{1}{2}U\overline{\theta^2})$, and arrive at the comparatively simple equations,

$$\overline{uw}\frac{\partial U}{\partial z} = \frac{g}{T}\overline{\theta w} - \epsilon, \tag{2.5}$$

$$\overline{\theta w}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right) = \kappa\overline{\theta\nabla^2\theta} - \beta\overline{\theta^2}. \tag{2.6}$$

It will be noticed that the radiative properties of the fluid do not enter directly into the equations for the turbulent energy and the convective heat transfer enters only through the buoyancy term $(g/T)\overline{\theta w}$. It is a plausible assumption that the general nature of the turbulent motion, i.e. its geometrical properties but not its intensity or scale, depends only on the relative magnitude of the buoyancy term, that is, on the flux Richardson number

$$R_f = \frac{g}{T}\overline{\theta w} \Big/ \overline{uw}\frac{\partial U}{\partial z}, \tag{2.7}$$

which is the ratio of the rate of loss of energy through working against buoyancy forces to the generation by working against Reynolds stresses. Since energy dissipation is essentially positive, the flux Richardson number must always be less than one.

The equation for the mean square temperature fluctuation (2.6) is a relation between the velocity field and the temperature field produced by its interaction with a gradient of mean temperature. Appreciable radiative heat transfer leads to added destruction of temperature fluctuations from the mean, to a lower mean square temperature fluctuation and to a lower absolute value of the convective heat transfer, $\overline{\theta w}$. Whether the gradient of potential temperature is positive or negative (stable or unstable to convective disturbances), the effect of radiative heat transfer is to reduce convective heat transfer in a given velocity field and mean temperature gradient and so to reduce the contribution (negative or positive) to the kinetic energy of the motion. If the flow is convectively unstable, radiative transfer reduces the instability as Goody (1956) found in his investigation of the effect of radiative transfer on convective instability between parallel horizontal planes. If the flow is stable, radiative transfer reduces the stability. Indeed, if radiative transfer were infinitely rapid, no motion of the fluid could cause any departure of local temperature from the mean value appropriate to the position in the flow and no buoyancy forces could be generated.

Equations (2.5) and (2.6) contain terms representing the rate at which viscosity converts turbulent kinetic energy to heat and the rate at which conductivity destroys temperature fluctuations from the mean. In fully turbulent flows, these rates are known to be independent of the actual values of the viscosity and the conductivity, and are determined by the large-scale characteristics of the turbulent motion, that is, by the intensity and scale of the motion (Townsend 1956). This may be expressed by writing these rates as

$$\epsilon = (\overline{w^2})^{3/2} L_\epsilon^{-1} \tag{2.8}$$

and

$$-\kappa\overline{\theta\nabla^2\theta} = \tfrac{1}{3}\overline{\theta^2}(\overline{w^2})^{1/2} L_\theta^{-1}, \tag{2.9}$$

where L_ϵ, L_θ are defined by these equations but are known to be nearly equal to the integral scale of the turbulence*.

It is now possible to obtain solutions of equations (2.5) and (2.6), expressing the convective heat transport and the Reynolds stress in terms of the gradients of mean velocity and mean temperature, the logarithmic cooling rate by radiation β, the dissipation lengths L_ϵ and L_θ, and the two correlation factors,

$$k_u = \frac{|\overline{uw}|}{\overline{w^2}}, \tag{2.10}$$

$$k_\theta = \frac{|\overline{w\theta}|}{[\overline{w^2}\,\overline{\theta^2}]^{1/2}}. \tag{2.11}$$

The most useful form relates the ordinary Richardson number

$$R_i = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)\bigg/\left(\frac{\partial U}{\partial z}\right)^2 \tag{2.12}$$

to the flux Richardson number, and is

$$R_f = \frac{H}{2}\left[1 - \left(1 - 12\,\frac{L_\theta k_\theta^2 R_i}{L_\epsilon k_u^2 H^2}\right)^{1/2}\right], \tag{2.13}$$

where

$$H = 1 + \frac{3L_\theta\beta}{k_u L_\epsilon\,|\partial U/\partial z|}. \tag{2.14}$$

It should be emphasized that the validity of equation (2.13) depends on only one assumption, that the effects of mean flow advection can be ignored, and that the quantities occurring in it are not local values but values averaged over a whole section of the flow. Its physical importance depends on the meaning given to these average values by the effective homogeneity of the flow and on the making of plausible assumptions about the variations of the non-dimensional ratios, k_u, k_θ and L_ϵ/L_θ, with stability.

3. Conditions for the maintenance of turbulent motion

From equation (2.13), we see that real values of the flux Richardson number are only possible if

$$R_i \leqslant \frac{H^2 k_u L_\epsilon}{12 k_\theta L_\theta}. \tag{3.1}$$

* The factor $\frac{1}{3}$ in (2.9) appears since, in any flow, the ratios of the rates of destruction to the respective intensities are roughly equal, i.e.

$$\frac{-\kappa\overline{\theta\nabla^2\theta}}{\overline{\theta^2}} \doteqdot \frac{\epsilon}{\overline{q^2}} \doteqdot \frac{\epsilon}{3\overline{w^2}}$$

Assuming that k_u, k_θ and L_ϵ/L_θ do not vary greatly with stability of the flow (reasons for supposing this in *fully* turbulent flow are given in Townsend 1958), this is equivalent to

$$R_f \leqslant \tfrac{1}{2}H. \tag{3.2}$$

These are limits set by non-existence of physical solutions of equations (2.5) and (2.6) for Richardson numbers which do not satisfy them, and are additional to the limit set by the energy equation alone,

$$R_f \leqslant 1. \tag{3.3}$$

For weak radiative transfer ($H < 2$, $\beta < \frac{1}{3}k_u(L_\epsilon/L_\theta)|\partial U/\partial z|$), the limit expressed by (3.1) and (3.2) applies, while for strong radiative transfer, the limit is given by equation (3.3).

The physical situation that leads to a double restriction on the possibility of turbulent flow may be made clearer by considering the equation for the turbulent kinetic energy

$$-\overline{uw}\,\frac{\partial U}{\partial z} = -\frac{g}{T}\,\overline{\theta w} + \frac{(\overline{w^2})^{3/2}}{L_\epsilon}, \tag{3.4}$$

and, in particular, the variation of the terms with assumed turbulent intensity for given gradients of mean velocity and mean temperature. An intermediate step in the algebra leading to the relation between the two forms of the Richardson number is (Townsend 1958)

$$\overline{\theta w} = \frac{-3k_\theta^2 L_\theta(\overline{w^2})^{1/2}(\partial T/\partial z + g/c_p)}{1+3\beta L_\theta(\overline{w^2})^{-1/2}}, \tag{3.5}$$

and so the energy equation may be written as

$$k_u\overline{w^2}\left|\frac{\partial U}{\partial z}\right| = \frac{3(g/T)k_\theta^2 L_\theta(\overline{w^2})^{1/2}(\partial T/\partial z + g/c_p)}{1+3\beta L_\theta(\overline{w^2})^{-1/2}} + \frac{(\overline{w^2})^{3/2}}{L_\epsilon}. \tag{3.6}$$

Consider now the way in which the two terms on the right, representing work done against buoyancy forces and loss of energy by dissipative processes, vary with turbulent intensity. This is done most conveniently by considering the ratio of their sum to the term on the left, the rate of production of turbulent energy by working against the Reynolds stresses. In figure 1, the variation of these ratios with $(\overline{w^2})^{1/2}/k_u L_\epsilon|\partial U/\partial z|$) is shown for negligible radiative transfer and various values of the Richardson number, R_i. If a stable value of the turbulent intensity exists, it must satisfy the energy equation, i.e. the sum of the ratios must equal one, and the total rate of energy loss must increase with turbulent intensity. It is clear from the diagram (i) that the sum of the ratios always has a minimum value, (ii) that the minimum value will exceed one in strongly stable flows, (iii) that the turbulent intensity does not become zero as the critical condition is approached, and (iv) that, in the critical flow, the flux Richardson number (which is the ratio of the work done against buoyancy forces to the energy production by shear) is less than one. This behaviour is typical of flows with weak radiative transfer and the condition (3.1) for turbulent flow is simply the condition that the minimum value of the sum of the ratios should be one or less.

In figure 2, the sum of the ratios is shown as a function of turbulent intensity for critical conditions and increasing radiative transfer. The

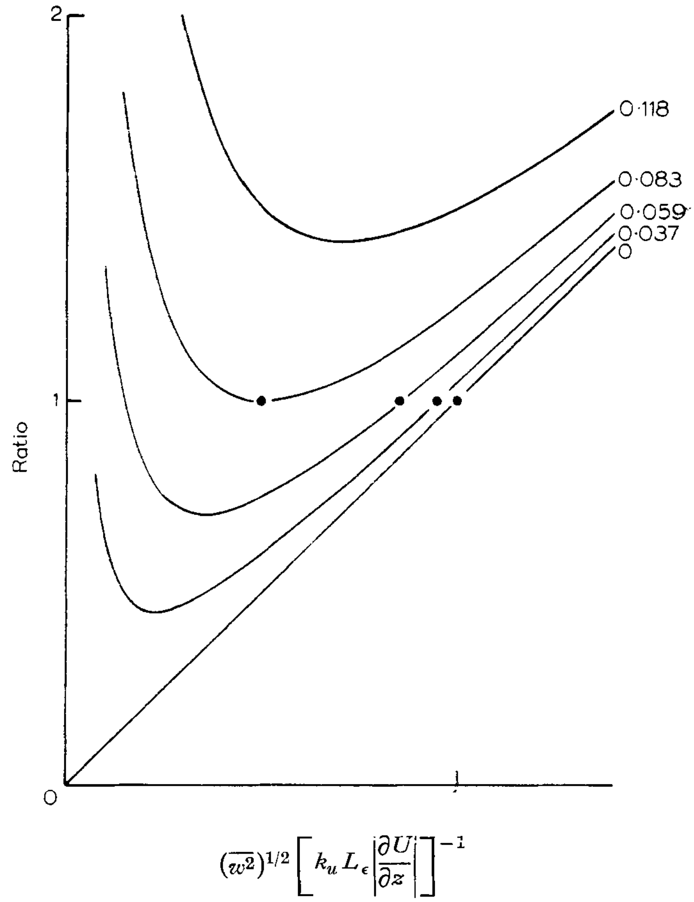

Figure 1. Variation with assumed turbulent intensity of the ratio of the total rate of energy loss to the rate of gain from the mean flow (no radiative transfer). (The numbers opposite the curves are values of $(L_\theta k_\theta^2/L_u k_u^2)R_i$, which in the text is equated with the Richardson number. The marked points indicate the stable configuration (if any) of the turbulent motion for the Richardson number concerned.)

position of the minimum moves to smaller values of the intensity and becomes zero when $\beta = \frac{1}{3}k_u(L_u/L_\theta)|\partial U/\partial z|$. For more intense transfer, the sum of the ratios always increases with turbulent intensity and the minimum possible value occurs at zero intensity.

The alternative conditions correspond with the suppression of the turbulent motion in two distinct ways. In conditions of weak radiative transfer, loss of energy through buoyancy forces is the dominant dissipative process for low values of the turbulent intensity as ordinary turbulent dissipation is for high values. In the critical condition, the intensity is neither so low that buoyancy forces could destroy the motion nor so high

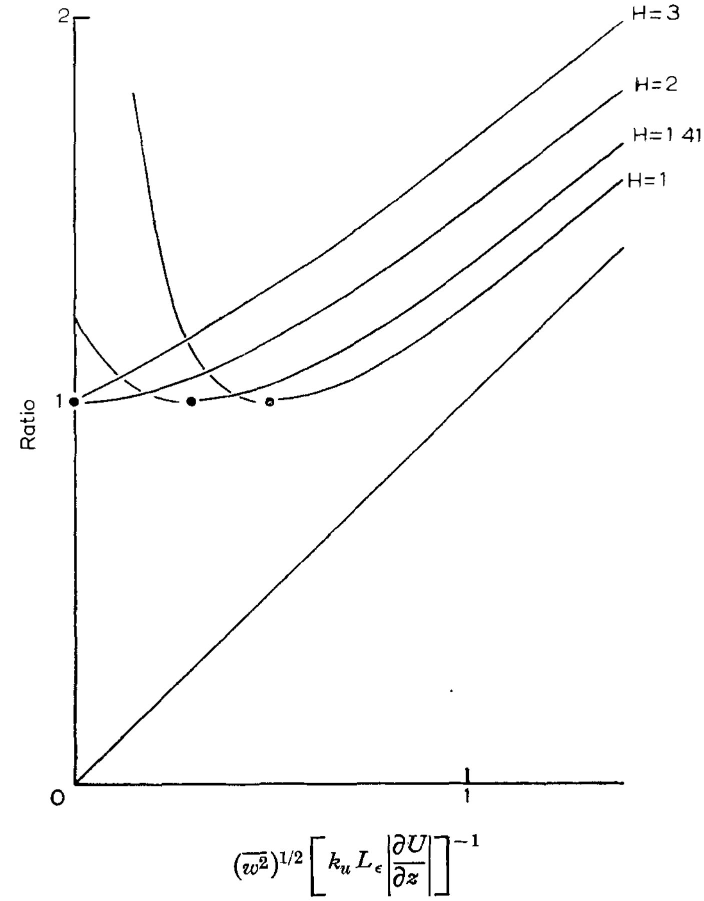

Figure 2. Variation with assumed turbulent intensity of the ratio of the total rate of energy loss to the rate of gain from the mean flow for the critical condition with various amounts of radiative transfer. (The marked points indicate the configuration and intensity of the turbulence in the critical condition for the various values of the radiation parameter *H*.)

that turbulent dissipation would, but has an intermediate value. As the limit is passed a sudden collapse of the turbulent motion will occur. Since the effect of radiative transfer is to destroy temperature fluctuations and to reduce buoyancy effects, the loss of energy through buoyancy forces is reduced, particularly at low intensities, and for sufficiently intense radiative transfer the critical condition has zero intensity. This corresponds with a condition of 'just no turbulence'.

The criterion for the maintenance of turbulent motion thus takes two forms, depending on the magnitude of H. The first is

$$(R_f)_{\text{crit.}} = \tfrac{1}{2}H, \qquad (R_i)_{\text{crit.}} = \frac{H^2 k_u^2 L_\epsilon}{12 k_\theta^2 L_\theta}$$

for $H < 2$ or

$$\frac{\beta}{|\partial U/\partial z|} < \frac{k_u}{3}\frac{L_\epsilon}{L_\theta} \doteqdot \tfrac{1}{6}.$$

The second, appropriate to conditions of strong radiative transfer, is

$$(R_f)_{\text{crit.}} = 1, \qquad (R_i)_{\text{crit.}} = \frac{k_u}{k_\theta^2}\frac{\beta}{|\partial U/\partial z|}$$

for $H > 2$ or

$$\frac{\beta}{|\partial U/\partial z|} > \frac{k_u}{3}\frac{L_\epsilon}{L_\theta} \doteqdot \tfrac{1}{6}.$$

Obviously the Richardson number is not a convenient parameter for describing flows with strong radiative transfer, and these last conditions are better expressed as

$$\left[\frac{\dfrac{g}{T}\left(\dfrac{\partial T}{\partial z} + \dfrac{g}{c_p}\right)}{\beta\left|\dfrac{\partial U}{\partial z}\right|}\right]_{\text{crit.}} = \frac{k_u}{k_\theta^2} \doteqdot 2$$

for

$$\beta\Big/\left|\frac{\partial U}{\partial z}\right| > \frac{k_u}{3}\frac{L_\epsilon}{L_\theta}.$$

4. The effect of radiative heat transfer on the temperature fluctuations

In the previous sections, the quantity H (defined by equation (2.11)) appears prominently and it is a measure of the ratio of the logarithmic rate of cooling by radiation β (equation (2.4)) to the mean rate of shear. To compute β, we need to know something about the dependence of $\mathscr{R}$, the net rate of heat gain per unit volume by radiative transfer, on the temperature distribution in the fluid. In general, $\mathscr{R}$ is a complicated expression depending on the difference between the temperature radiation from the fluid volume and its rate of absorption of radiation from other parts of the fluid and from outside the fluid, but our present concern is only with fluctuations of temperature and so we do not need to know the nature or quantity of the constant radiation from outside the flow. To calculate the radiation from other parts of the fluid, we assume that the gas is everywhere close to thermodynamic equilibrium and that we may neglect the effects of unmodified scattering.

The equation of radiative transfer is (Chandrasekhar 1950)

$$-\frac{1}{k_\nu \rho}\frac{dI_\nu}{dr} = I_\nu - B_\nu(T), \tag{4.1}$$

where k_ν is the mass absorption coefficient for radiation of frequency ν, I_ν is the specific intensity, i.e. the flux of radiant energy per unit solid angle per unit frequency interval, r is distance along the direction of propagation, and $B_\nu(T)$ is the Planck function for black-body emission. This equation may be integrated to give the intensity at a point as

$$I_\nu = \int_0^\infty B_\nu(T_r) e^{-k_\nu \rho r} k_\nu \rho \, dr, \tag{4.2}$$

where r is the distance between the point concerned and the moving point of integration, and the density is assumed to be substantially constant. The total intensity from all parts of the fluid is then

$$\int I_\nu \, dw = \int B_\nu(T_r) e^{-k_\nu \rho r} \frac{k_\nu}{r^2} \rho \, dV(\mathbf{r}), \tag{4.3}$$

where the integrals extend respectively over all solid angles and all the fluid, the subscript r denotes values at the element of integration, and $V(\mathbf{r})$ is an element of volume at the position $\mathbf{r}$. From this it follows that absorption of temperature radiation from the surrounding fluid proceeds at a rate per unit mass of

$$\int \int_0^\infty B_\nu(T_r) k_\nu e^{-k_\nu \rho r} \frac{1}{r^2} \rho \, dV(\mathbf{r}) d\nu,$$

or, more concisely, of

$$-\frac{1}{\pi}\int \frac{\partial^2 E(T_r, \rho r)}{\partial(\rho r)^2} \sigma T_r^4 \frac{\rho}{r^2} \, dV(\mathbf{r}), \tag{4.4}$$

where

$$E(T, \rho r) = \int_0^\infty B_\nu(T)(1 - e^{-k_\nu \rho r}) \, d\nu \Big/ \int_0^\infty B_\nu(T) \, d\nu$$

is the emissivity of the fluid. Heat radiation from an element of temperature T proceeds at a rate

$$4\bar{k}(T)\sigma T^4, \tag{4.5}$$

where

$$\bar{k}(T) = \int_0^\infty k_\nu B_\nu(T) \, d\nu \Big/ \int_0^\infty B_\nu(T) \, d\nu = \left[\frac{\partial E(T, \rho r)}{\partial(\rho r)}\right]_{\rho r = 0}.$$

Both $E(T, \rho r)$ and $\bar{k}(T)$ depend on temperature, but if the larger values of k_ν are distributed fairly evenly within the range of the Planck function, $B_\nu(T)$, both will be slowly varying functions of temperature.

Neglecting the variations of $E(T, \rho r)$ and $\bar{k}(T)$ with temperature, the net rate of temperature rise by internal radiative transfer is

$$-\frac{\sigma}{\pi c_p}\int [(T_r + \theta_r)^4 - (T + \theta)^4] \frac{E''(\rho r)}{r^2} \rho \, dV(\mathbf{r}), \tag{4.6}$$

where dashes denote differentiation of $E(\rho r)$ with respect to ρr (note that $E''(\rho r)$ is essentially negative).

If the expression (4.6) is multiplied by the temperature fluctuation and the mean values taken, the result is the radiation term in equation (2.4) which represents the rate of destruction of $\frac{1}{2}\overline{\theta^2}$ by radiative transfer. Assuming that $|\theta| \ll T$, this gives

$$-\beta\overline{\theta^2} = \frac{\overline{\mathscr{R}\theta}}{\rho c_p} = -\frac{\sigma}{\pi c_p}\int \overline{[(T_r+\theta_r)^4-(T+\theta)^4]\theta}\,\frac{E''(\rho r)}{r^2}\,\rho\,dV(\mathbf{r}). \quad (4.7)$$

It is probable that the temperature covariance $\overline{\theta(\mathbf{x})\theta(\mathbf{x}+\mathbf{r})}$ is nearly symmetrical about a plane perpendicular to the gradient of mean temperature, and then

$$-\beta\overline{\theta^2} = -\frac{4\sigma T^3}{\pi c_p}\int [\overline{\theta\theta_r}-\overline{\theta^2}]\,\frac{E''(\rho r)}{r^2}\,\rho\,dV(\mathbf{r}), \quad (4.8)$$

indicating that the destruction of $\frac{1}{2}\overline{\theta^2}$ by radiative transfer takes place at a rate whose maximum value in a fluid of given properties is

$$-\frac{4\sigma T^3}{\pi c_p}\,\overline{\theta^2}\int \frac{E''(\rho r)}{r^2}\,\rho\,dV(\mathbf{r}) = 16\,\frac{\sigma T^3}{c_p}\,\bar{k}\,\overline{\theta^2}. \quad (4.9)$$

This maximum value occurs when the scale of the temperature fluctuations is so small that $\overline{\theta\theta_r}$ becomes negligible before the transmission is sensibly different from one*. In general, the rate of destruction of $\frac{1}{2}\overline{\theta^2}$ is

$$\beta\overline{\theta^2} = 16F\bar{k}\,\frac{\sigma T^3}{c_p}\,\overline{\theta^2} \quad (4.10)$$

where

$$F = -\frac{1}{4\pi}\int\left(1-\frac{\overline{\theta\theta_r}}{\overline{\theta^2}}\right)\frac{E''(\rho r)}{\bar{k}r^2}\,\rho\,dV(\mathbf{r})$$

$$= 1+\frac{1}{4\pi}\int\frac{\overline{\theta\theta_r}}{\overline{\theta^2}}\,\frac{E''(\rho r)}{\bar{k}r^2}\,\rho\,dV(\mathbf{r}). \quad (4.11)$$

Defining the average value of $\overline{\theta\theta_r}/\overline{\theta^2}$ over a spherical shell of radius r as

$$Q(r) = \frac{1}{4\pi r^2}\int \frac{\overline{\theta\theta_r}}{\overline{\theta^2}}\,dS(\mathbf{r}) \quad (4.12)$$

($dS(\mathbf{r})$ is an area element of the spherical shell), we find that

$$F = 1+\int_0^\infty Q(r)\,\frac{E''(\rho r)}{\bar{k}}\,\rho\,dr, \quad (4.13)$$

showing how F depends on the relative extents of the temperature correlation function $Q(r)$ and the transmission coefficient $\Pi(\rho r) = E''(\rho r)/E''(0)$.

For two limiting conditions, the factor F may be expressed more simply.

(a) If the fluid is effectively transparent, i.e. if

$$\int_0^\infty \Pi(\rho r)\,dt = -\frac{1}{\rho}\,\frac{\bar{k}}{E''(0)} \gg \int_0^\infty Q(r)\,dr,$$

* In this event, the fluid is effectively transparent, radiation from other parts of the fluid is not absorbed, and equation (4.9) is a direct consequence of (4.5).

then

$$F = 1 + \frac{\rho E''(0)}{\bar{k}} \int_0^\infty Q(r)\, dr. \tag{4.14}$$

(*b*) If the fluid is nearly opaque, i.e. if

$$\int_0^\infty \Pi(\rho r)\, dr \ll \int_0^\infty Q(r)\, dr,$$

only small values of r for which $Q(r)$ is nearly one contribute to the integral. For these values,

$$Q(r) = 1 - \tfrac{1}{6}r^2 \overline{\left(\frac{\partial\theta}{\partial x_i}\right)^2} \Big/ \overline{\theta^2},$$

and so

$$F = -\frac{E''(0)M^3}{6\bar{k}\rho} \overline{\left(\frac{\partial\theta}{\partial x_i}\right)^2} \Big/ \overline{\theta^2}, \tag{4.15}$$

where

$$M^3 = \int_0^\infty u^2 \Pi(u)\, du.$$

It may be noticed that the actual rate of destruction of temperature fluctuations by radiative transfer is

$$-\frac{8\sigma T^3 E''(0)}{3\rho^2 c_p} M^3 \overline{\left(\frac{\partial\theta}{\partial x_i}\right)^2},$$

equivalent to an additional 'radiation' heat diffusivity of

$$k_R = -\frac{8\sigma T^3 E''(0)}{3\rho^2 c_p} M^3. \tag{4.16}$$

This was the same diffusivity as was found by Goody (1956).

5. Application to motion in the atmosphere

This analysis is expected to apply to flows at a sufficient distance from solid boundaries to be substantially unaffected by their presence, a condition which would be satisfied in the atmosphere and outside the earth's boundary layer. Within the boundary layer, the theory might be capable of representing the order of magnitude of the effect of radiation on the turbulent motion, but the essential inhomogeneity of the motion would make definition of flow averages very uncertain.

The logarithmic cooling rate, whose ratio to the mean velocity gradient determines the influence of radiative transfer on the motion, is a function of the scale of the temperature fluctuations as well as the radiative properties of the atmosphere unless the atmosphere is effectively transparent to temperature radiation of the gas. The condition for this is that the absorption length

$$L = \int_0^\infty \Pi(\rho r)\, dr = 1/\rho \int_0^\infty k_\nu B_\nu\, d\nu \Big/ \int_0^\infty k_\nu^2 B_\nu\, d\nu \tag{5.1}$$

should be large compared with the scale of the temperature fluctuations, which must be roughly equal to the scale of the mean velocity variation or the vertical extent of the flow. For a line absorption spectrum,

$$L \doteqdot \frac{2}{\rho k_m},$$

where k_m is the maximum value of the absorption coefficient and may be very much larger than $\bar{k}$, the ratio depending on the ratio of line spacing to line width. For air of sea-level density containing 2% by volume of water-vapour,

$$L \doteqdot 400 \frac{\bar{k}}{k_m} \text{ cm},$$

and only motions of very small scale could be described by the 'transparent' approximation. For air of density $10^{-7}\,\text{gm}\,\text{cm}^{-3}$, which occurs at heights around 70 km, containing $2{\cdot}5 \times 10^{-4}$ by volume of carbon dioxide,

$$L \doteqdot 10^4 \frac{\bar{k}}{k_m} \text{ km},$$

and a possibility of using the transparent approximation exists.

From information given by Elsasser (1942) and by Curtis & Goody (1956), the mean absorption coefficient at 300° K is

$$\bar{k} = (90q_w + 150q_c + 215q_z)\,\text{gm}^{-1}\,\text{cm}^2 \tag{5.2}$$

where q_w is the proportion by volume of water-vapour, q_c is the proportion by volume of carbon dioxide, and q_z is the proportion by volume of ozone. Using equation (4.10), the logarithmic cooling rate in transparent conditions at 300° K is

$$\beta = (0{\cdot}21q_w + 0{\cdot}35q_c + 0{\cdot}51q_z)\,\text{sec}^{-1}. \tag{5.3}$$

(The rate at other temperatures can be estimated by neglecting the variation of $\bar{k}$ with temperature and supposing β to vary as the cube of the absolute temperature.) In table 1, the gradients of potential temperature, $\partial T/\partial z + g/c_p$, that are just sufficient to allow turbulent motion are listed for three values of the mean velocity gradient and for four values of the logarithmic cooling rate. These values refer to (i) a non-radiating atmosphere, (ii) one containing $2{\cdot}5 \times 10^{-4}$ of carbon dioxide, (iii) one containing $2{\cdot}5 \times 10^{-4}$ of carbon dioxide and 2×10^{-3} of water-vapour, (iv) one containing $2{\cdot}5 \times 10^{-4}$ of carbon dioxide and 2×10^{-2} of water-vapour, all effectively transparent and at 300° K. For the least absorbing atmosphere, the critical temperature gradient is very little different from that for a non-absorbing atmosphere, but for the most absorbing atmosphere, the critical gradient must be an order of magnitude greater. The purpose of the table is to show the order of magnitude of radiation effects in air of various compositions and not to assert that air of these compositions exists at great heights. The relevance (or otherwise) of these assumed compositions may be explained very briefly. The assumed content of carbon dioxide is about that known to exist in the stratosphere, and the two assumed contents of water-vapour have no more justification than (i) that these concentrations could exist without condensation at an air density of 10^{-7} gm cm^{-3} and any likely temperature, (ii) that the occasional presence of water-vapour in significant quantities may be indicated by the occurrence of noctilucent clouds at great heights, and (iii) that these compositions lead to values of β suitably spaced for a table intended to be illustrative rather than definitive.

β (sec^{-1})	Critical gradients of potential temperature (deg. km^{-1})		
	$\left\|\frac{\partial U}{\partial z}\right\| = 0{\cdot}005$ sec^{-1}	$\left\|\frac{\partial U}{\partial z}\right\| = 0{\cdot}01$ sec^{-1}	$\left\|\frac{\partial U}{\partial z}\right\| = 0{\cdot}02$ sec^{-1}
0	0·064	0·255	1·02
$0{\cdot}9 \times 10^{-4}$	0·079	0·284	1·07
$5{\cdot}1 \times 10^{-4}$	0·166	0·435	1·36
$4{\cdot}3 \times 10^{-3}$	1·32	2·63	5·26

Table 1.

The failure of the transparent approximation in the lower atmosphere makes it desirable to obtain an estimate of the logarithmic cooling rate for intermediate conditions between transparency and opacity. The general expression for the factor F (equation (4.13)) may be integrated by parts to give

$$F = \int_0^\infty \frac{E'(\rho r)}{\bar{k}} \frac{dQ(r)}{dr}\, dr \tag{5.4}$$

showing that F is the mean value of $E'(\rho r)/\bar{k}$, taken over all values of r with a weighting factor $dQ(r)/dr$. It is characteristic of line absorption that the initial, very rapid decrease of $E'(\rho r)$ is followed by an extended region of very slow decrease, while the weighting factor $dQ(r)/dr$ is small near $r = 0$. It follows that if we define a length L_s by

$$F = E'(\rho L_s)/\bar{k}, \tag{5.5}$$

it will be roughly equal to the scale of the temperature fluctuations. On this basis, critical gradients of potential temperature have been calculated for flow in air containing $2^0/_0$ of water-vapour at sea-level density for a range of scales and velocity gradients (table 2). For this atmospheric composition,

$$F = 3{\cdot}7 L_s^{-1/2} \qquad (L_s \text{ in cm}) \tag{5.6}$$

using results quoted by Elsasser (1942). Except for the largest scale of 1 km, substantial effects of radiative transfer are predicted for this composition and mean velocity gradients of the kind that might occur at heights over 20 metres.

	Critical gradients of potential temperature (deg. km^{-1})		
	$\left\|\frac{\partial U}{\partial z}\right\| = 0{\cdot}005\ \text{sec}^{-1}$	$\left\|\frac{\partial U}{\partial z}\right\| = 0{\cdot}01\ \text{sec}^{-1}$	$\left\|\frac{\partial U}{\partial z}\right\| = 0{\cdot}02\ \text{sec}^{-1}$
$\beta = 0$	0·064	0·255	1·02
$L_s = 10^2$ cm, $F = 0{\cdot}37$	0·49	0·98	2·23
$L_s = 10^3$ cm, $F = 0{\cdot}12$	0·171	0·445	1·37
$L_s = 10^4$ cm, $F = 0{\cdot}037$	0·092	0·305	1·12
$L_s = 10^5$ cm, $F = 0{\cdot}012$	0·072	0·270	1·05

Table 2.

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